Upper Bound on Cell Size for Moment-Method Solutions

MARK J. HAGMANN, STUDENT MEMBER, IEEE, OM P. GANDHI, SENIOR MEMBER, IEEE, AND CARL H. DURNEY, MEMBER, IEEE

Abstract—**When pulse functions are used in moment-method solutions, failure to allow for variation of the field within each cell limits the maximum usable electrical size of the cells. Appreciable error is expected for $|k|l \geq 2$ in one or two dimensions and $|k|l \geq \sqrt{6}$ in the three-dimensional problem where l is the side of a cell and k is the propagation constant in the material.**

I. Introduction

In electromagnetics, discretization for transformation of an integral equation to a matrix equation is often accomplished using pulse functions as a basis [1]–[4]. When using pulse functions, the scatterer is partitioned into a number of cells N, where N is large enough that complex permittivity and the complex time-independent electric field may be assumed constant within each subvolume.

In a given problem, it is common to estimate the maximum usable cell size and try several values of N to test for apparent convergence of the resulting solutions [5]. An approximate upper bound on cell size may be found from observation of the oscillatory nature of the kernel in the integral equation, which suggests that the cell size not exceed $\lambda_0/5$, where λ_0 is the free space wavelength [6]. It is the object of this paper to establish a significantly tighter upper bound on cell size in moment-method solutions with scatterers when pulse functions are used as the basis.

II. Evaluation of the Bound on Cell Size

Consider a source-free region of space in which the dielectric properties are homogeneous, linear, and isotropic. All fields are assumed to have exp $(j\omega t)$ time variation. We may set up a local Cartesian coordinate system at any point in the region and require that the homogeneous wave equation be satisfied:

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0. \tag{1}$$

Six points are chosen a distance S from the origin on halves of each of the three local axes. f_i will represent the value of one component of $\vec{E}$, say E_x, at the ith point. Let

$$\delta f_i \equiv f_i - f_o \tag{2}$$

where f_o is the value of E_x at the origin.

If S is small enough so that there is little variation in E_x, the difference equation approximation of (1) may be used with the result

$$\sum_{i=1}^{6} \delta f_i + k^2 S^2 f_o = 0. \tag{3}$$

The greatest (in absolute value) of the δf_i must satisfy

$$\frac{|\delta f_i|}{|f_i|} \geq \frac{|k|^2 S^2}{6}. \tag{4}$$

When (4) predicts large variations, we cannot expect the prediction to be quantitative, but we may safely infer that substantial variation of the fields will always occur within the volume of a cell containing the six points if the predicted fractional variation has a value of, say, one-half, for which

$$|k|S = \sqrt{3}. \tag{5}$$

Restricting our attention to cubical cells, we note that the smallest cube containing the six points has side

$$l = \sqrt{2}S \tag{6}$$

so large variations in the fields are expected for

$$|k|l \geq \sqrt{6}. \tag{7}$$

The corresponding result for square cells with side l in the two-dimensional problem or linear cells of length l in the one-dimensional problem is

$$|k|l \geq 2. \tag{8}$$

In an efficient moment-method solution such as in [1]–[4], variation of the Green's function within each cell is closely approximated so that the primary source of error is imperfect representation of the fields by the basis. If pulse functions are used, it is assumed that the fields are constant within each cell. Then appreciable error is expected if (8) is satisfied in a one- or two-dimensional problem or if (7) is satisfied in a three-dimensional problem. The smoothing property of the integral operator causes the error in the solution to be somewhat less than may be anticipated for a simple basis, but we may still expect that (7) and (8) give a reasonable upper bound for cell size.

III. Examples

The bound of (7) is tighter than that found from consideration of the oscillatory nature of the kernel for scatterers having a relative permittivity ε_r such that $|\varepsilon_r| > 3.8$. In order to illustrate this, a couple of biological applications using dielectric properties of muscle [7] will be considered

Manuscript received November 29, 1976; revised April 4, 1977. This work was supported by the U.S. Army Medical Research and Development Command, Washington, DC, under Contract DAMD 17-74-C-4092.

The authors are with the Department of Electrical Engineering, University of Utah, Salt Lake City, UT 84112.

TABLE I
VALUES OF THE INDUCED ELECTRIC FIELD AT THE CENTER OF MUSCLE CUBES AT 2450 MHz AS GIVEN IN [4]: $\varepsilon_1/\varepsilon_0 = 47.0$, $\sigma = 2.21$ mho/m

	Number of Cells	ℓ/λ_ε	$\lvert k\rvert\ell$	ℓ/λ_0	$\lvert E\rvert$ Center (V/m)
Cube one	1	1	6.37	0.144	0.0789
$a = \lambda$	27	1/3	2.12	0.0481	0.0922
Cube two	1	1/4	1.59	0.0361	0.0592
$a = \lambda/4$	27	1/12	0.531	0.0120	0.0556

TABLE II
ERRORS IN THE AVERAGE SPECIFIC ABSORBED POWER DENSITY FOR AN INFINITE CYLINDER OF MUSCLE WITH A 20- BY 20-cm CROSS SECTION AT 100 MHz: $\varepsilon_1/\varepsilon_0 = 68.0$, $\sigma = 0.890$ mho/m

Number of Cells	ℓ/λ_ε	$\lvert k\rvert\ell$	ℓ/λ_0	Error
4	0.367	2.76	0.0334	-34.90%
9	0.245	1.84	0.0222	-16.40%
16	0.183	1.38	0.0167	-8.64%
25	0.147	1.11	0.0133	-4.96%

since the complex permittivity has a relatively large magnitude.

Table I uses the results given in [4] for two muscle cubes having sides of $a = \lambda_\varepsilon$ and $\lambda_\varepsilon/4$, respectively, at 2450 MHz, where λ_ε is the wavelength in the material. No analytical solution is available for comparison, but variation of the calculated electric field with cell size suggests that there is significant error in the single-cell solution for the larger muscle cube. The small values of l/λ_0 suggest that the oscillatory nature of the kernel should contribute little error in the four calculations. Values of $|k|l$ and (7) suggest that significant error should be found using one cell with the larger cube, as is observed.

For the second example, Richmond's method [1] was used to calculate the average specific absorbed power density in an infinite cylinder of muscle with a 20- × 20-cm square cross section. A plane wave at 100 MHz with a power density of 1 mW/cm^2 incident normal to one of the four congruent flat surfaces was used for TM excitation. The average specific absorbed power density found using 100 cells is 0.05534 mW/cm^3, which was used as a standard, differing by 0.39 percent from the value found using 81 cells. Table II gives the results found using fewer cells. Note that significant error is encountered when (8) is satisfied even though l/λ_0 is so small that the oscillatory nature of the kernel contributes negligible error.

IV. CONCLUSIONS

When pulse functions are used in moment-method solutions, failure to allow for variation of the field within each cell limits the maximum usable electrical size of the cells. Appreciable error is expected for $|k|l \geq 2$ in one or two dimensions, and $|k|l \geq \sqrt{6}$ in three-dimensional problems, where l is the side of a cell. The new upper bounds for cell size are significantly tighter than those found from the oscillatory nature of the kernel if the scatterer has a large relative permittivity. The new upper bounds have been demonstrated with two- and three-dimensional solutions.

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